3. LEVEL: MEDIUM

A very powerful tool is the recognition that, if two cells in a given row are each limited to the same two numbers, then that uses up these numbers for the row, and both numbers can be crossed out wherever else they occur in that row. If **a** and **b** represent digits, and square brackets [ ] indicate different cells, then the twofold ambiguity rule can be written as:

[**a b**] [**a b**] —> Neither **a** nor **b** can appear in any of the other seven cells of that row.

What has been said, and will be said, about *rows*, applies equally well but independently to *columns* and to *boxes*. For example, if one 3 x 3 box has two cells that each are limited to either a 5 or a 7, then 5 and 7 can be deleted from all the other seven cells of that box. The essential criterion for using these twofold ambiguities to cross out digits is that the two cells should be absolutely identical. Allow the possibility of a third digit in one of them, and the method fails. But the method of twofold ambiguities works independently for rows, for columns, and for boxes, making it very powerful.

Threefold ambiguity cancellations also are encountered frequently. If three digits, **a**, **b** and **c**, appear in some combinations in three different cells [ ] of a row (or column, or box), and if no other digit is allowed in these three cells, then these three cells must contain **a**, **b** and **c** in some as yet-undetermined order. Hence **a**, **b** and **c** can be crossed off as forbidden to the other six cells of the row (or column, or box). Examples of threefold ambiguities are:

[**a b c**] [**a b c**] [**a b c**] This one is less common but does turn up.
[**a b c**] [**a b c**] [**a b**]
[**a b c**] [**a b**] [**a c**]
[**a b**] [**b c**] [**c a**] This one is reasonably common.

But a fourth digit in one cell wipes out the method entirely, as:

[**a b d**] [**b c**] [**c a**] No conclusions can be drawn.

For if the first cell holds a **d**, then either **a** or **b** would be forced to show up in one of the remaining six cells in the row, and you couldn't generalize and cross out **a, b** and **c** from those cells.

Fourfold ambiguity cancellations are less common, but occur more often than you might think. If four different cells contain only two or more of a set of four digits, then these four digits can be deleted from the other five cells of that row, or column or box. It turns out after the fact that this strategy is a familiar one to Sudoku solvers. For example, Andrew C. Stuart in his "The Logic of Sudoku" (Michael Mepham, 2007) refers to twofold, threefold and fourfold ambiguities as naked pairs, naked triples and naked quads.

Let's try out the twofold ambiguity strategy with a Sudoku puzzle rated Medium, on sheet A (Plate 3.1). We will switch to the *Excel* form of the matrix because it is cleaner, but will continue with manual crossing-out for the moment. Row/column/box cancellation of all of these original numbers leads to sheet B (Plate 3.2), and this is where the puzzle really begins. Sheet B reaches an impasse at once using just the methods of the *Easy* puzzle. There are no initial single-digit cells at all. But use of twofold and higher ambiguities will lead to a solution. The path of analysis is listed below. (As before, whenever you assign a number to a cell, then immediately cancel that number everywhere it occurs in the row, column and box that contains that cell.)

1) No single-digit cells are present after initial crossouts are completed.
2) D6/D7=[3 5]. That is, both D6 and D7 constitute an ambiguous pair.
   Either cell can hold a 3 or a 5, but nothing else.
   Cancel all 3's and 5's elsewhere in column D only.
3) D2=6
4) A2/A5=[3 4]. Cancel other 3's and 4's from column A.
5) A9=8, A7=9, A1=6, D8=1, D5=2
   .....and column D is solved except for the [3 5] twofold ambiguity at D6/D7. But this ambiguity doesn't give us any new information because we already know the other seven cells in column D.

6) F3/H3=[3 5]. Cancel all other 3's and 5's in row 3.
7) E3=2, I3=6, B3=9
   .....and row 3 is solved except for the [3 5] ambiguity at F3/H3.

8) F4/F5=[3 4]. Cancel other 3's and 4's in column F and in box 5.
9) F3=5, so H3=3, breaking one ambiguity and solving row 3.
10) F8=8, E2=3, so F2=7
    .....and column F is solved except for the [3 4] ambiguity at F4/F5.

11) H4=4, so F4=3, F5=4, A5=3 and A2=4, breaking two ambiguities.
    .....and columns F and A are solved.

12) E5=1, I5=7, and row 5 is solved.

13) I1/I8=[4 5] ambiguity. Cancel other 4's and 5's in column I.
14) So I9=3, G9=4, G1=7
15) G8=9, G6=3, G4=6, and column G is solved.

16) I8=5, so I1=4, breaking the I1/I8 ambiguity and solving column I.

17) H1=5, C1=3, B1=2, completing row 1.
18) C2=5, B4=1, C4=9, completing rows 2 and 4.
19) C6=4, so B6=7

20) E8=4, so E7=5, completing row 8 and column E.
21) D7=3, D6=5, completing column D
22) B7=4, H7=7, completing row 7.
23) H6=8, B9=5, C9=1, H9=2
.....and the entire puzzle is solved, as shown in Plate 3.3.

Note that this puzzle would have been totally unsolvable were it not for the twofold ambiguity strategy, whereas the previous *Easy* puzzle didn't need it at all.

But these methods are still insufficient enough to solve a really *Hard* puzzle. Another strategy remains to be invoked, and it will be used next with a puzzle rated as *Hard*.

Plate 3.1: Initial digit set, A

Col: / Row:	A	B	C	D	E	F	G	H	I
1				9	8	1			
2		8					2	1	9
3	1		7	4			8		
4	5			8	7				2
5		6	8				5	9	
6	2				6	9			1
7			6			2	1		8
8	7	3	2					6	
9				7	9	6			

Plate 3.2: After initial crossouts, B

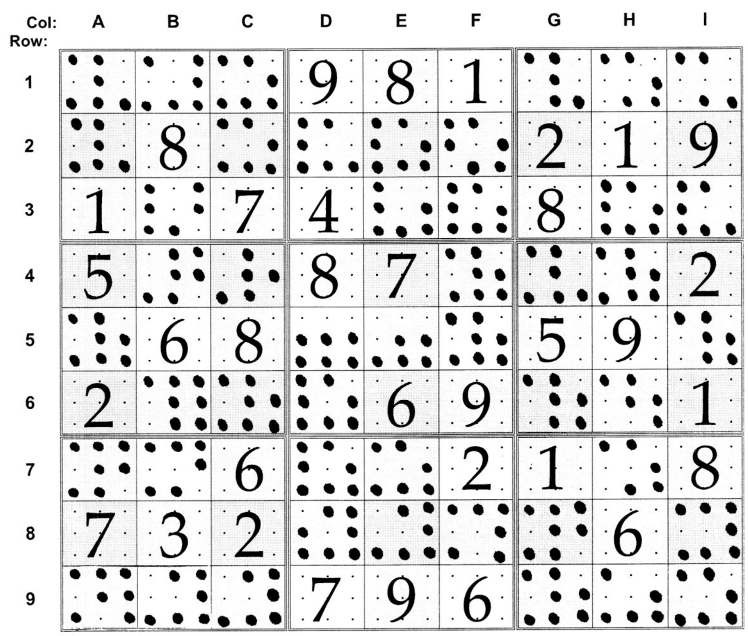

Plate 3.3: Final solution, C

Col: / Row:	A	B	C	D	E	F	G	H	I
1	6	2	3	9	8	1	7	5	4
2	4	8	5	6	3	7	2	1	9
3	1	9	7	4	2	5	8	3	6
4	5	1	9	8	7	3	6	4	2
5	3	6	8	2	1	4	5	9	7
6	2	7	4	5	6	9	3	8	1
7	9	4	6	3	5	2	1	7	8
8	7	3	2	1	4	8	9	6	5
9	8	5	1	7	9	6	4	2	3