6. BEYOND DIABOLICAL

Many years ago I heard the story of a small boy whose parents presented him with a toy drum for his birthday. He played the drum incessantly, and nearly drove his grandfather mad until the old man presented his grandson with a beautiful pearl-handled pocket knife, saying to him, "Here's a belated birthday present that you might like. And oh, by the way, do you know what is inside your drum?" This chapter is devoted to what's inside your Sudoku puzzle, and parallel analysis is the knife.

One of the hypnotic aspects of parallel analysis is that it allows you to study the effect of changing a particular digit, and even how to design your own Sudoku puzzle. I have found it especially interesting to choose a published puzzle that happens to have a digit in the central cell E5, and then to vary this digit and watch what this does to the solution process and to the answers obtained. Suppose we delete E5=6 from Plate 5.1. How many alternative digits could we substitute for it, and what kinds of solutions would result? That is the issue in this "Beyond Diabolical" chapter.

Of course one could simply solve the Sudoku independently eight times, setting the central digit in turn to 1, 2, 3, 4, 5, 7, 8 and 9. But that would be tedious and unnecessary. The first step is to find out which digits can possibly occupy cell E5 and which are disallowed because of clashes with the other starting digits. To find this out, set up a matrix with all the starting digits present except the one that you have deleted (Plate 6.1). When you carry out the initial cancellations of disallowed digits (Plate 6.2) you find that only 1, 4, 5, 6 and 7 can be tolerated in cell E5. (This is obvious since the other four digits form the corners of the box that contains E5.)

Next, proceed by the basic methods of Chapters 2–4 to eliminate as many other digits from E5 as you can. You can see from Plate 6.3 that E5=5 is ruled out since 5 is needed in cell E4 to complete row 4. And both 4 and 7 are eliminated since cells E2 and E6 form an ambiguous pair limited to exactly those two digits. All that is left for E5 is 1 or 6. The original puzzle in the previous chapter had E5=6, so in this chapter we need only to consider what happens if E5=1. This is a particularly simple case involving just two possible central digits. Other Sudokus can accommodate as many as five or more different digits in cell E5. This makes the analysis very complicated. Let us be glad that Plate 6.1 only has two choices.

A word of warning about the initial search leading from Plate 6.2 to 6.3. Imagine that cell E5 has a barrier around it that allows information to flow in, but not to flow out. If an outside cell has a particular digit in the same row, column or box as E5, that clearly eliminates this digit from the central cell. But the contents of E5 cannot help you make any decisions concerning the rest of the matrix, since at this stage you have not yet decided which of the possible E5 digits you will choose. For example, if the 4 had been knocked out of cell D5 in Plate 6.3 by some other cancellation, you could not regard D5 and E5 as constituting a twofold ambiguity and use them to eliminate 1 and 6 from F5.

With as many digits as possible disposed of, one is free to try each possible E5 occupant in turn, and see whether it leads to a solution. In our case, only E5=1 remains. With 1 in place, use the basic methods of Chapters 2–4 to eliminate as many digits as you can. All other 1's in row 5, column E and box 5 can be canceled (Plate 6.4), but then the puzzle stalls again. It is time for parallel analysis.

Which cell to choose first? I selected A2=[2,5] because it is one cell of a triplet. A2, C2 and C6 all are limited to either 2 or 5, and once you have specified the contents of one cell, you know all three. Gratifyingly, A2=2 led immediately to the solution **@a** on Plate 6.5. But the other possibility, A2=5, didn't crash or fail; it just stalled (Plate 6.6). An important rule to remember is that <u>a stall means that the game isn't over yet</u>.

It is a fundamental guiding principle when designing a Sudoku for public consumption that only <u>one</u> solution should exist. Sudokus with more than one solution are regarded by professional designers as degenerate or defective. But we aren't designing puzzles for the newspapers here; we are seeing what happens when we make radical changes in a published puzzle. So A2=2 is a satisfying result, but the game is not yet over. A2=5 is has only stalled, not crashed. Where it will lead when we continue is unknown, but is something that is fun to find out.

With A2=5 added to the matrix, I next chose D3=[2,6], again because it was paired with E3 and would deliver twice as much information for the money. A2=5 followed by D3=6 unexpectedly produced a second valid solution **@b** (Plate 6.7). It is interesting to compare this new solution with that in Plate 6.5. They resemble one another closely in places, but differ at 26 of their 81 positions. Some of these are simple exchanges of locations for two digits, but other differences are more complicated. These two equally valid answers would immediately cause a professional Sudoku designer to throw everything out and start over. But we are exploring rather than designing, so who cares what they think?

A2=5 plus D3=2 led to yet another stall (Plate 6.8). This stall again tells us, "Things aren't done yet; don't stop now." So I chose H5=[7,8], again because it was paired with another cell, H9. The results were really surprising. H5=7 led to two examples (Plates 6.9 and 6.10) of a new kind of solution that is so unusual that it merits a special symbol, #. This # will indicate a solution that contains an endless loop of 2-digit cells having different answers depending on your choice of one digit in the loop. The simplest case would be a rectangle of four cells, each of which contains either a 5 or a 9. Traveling clockwise around the loop, the solution could be either 5–9–5–9 or 9–5–9–5, and there is <u>nothing</u> to indicate that one answer is better than the other. I call these "ambiguous loops". Again, professional Sudoku designers avoid these like the plague, but I like them. We will come back to this issue in the next chapter.

In the present case, both Plates 6.9 and 6.10 exhibit the same strange 8-cell loop through the central vertical stack of boxes. Starting from the upper right corner in a clockwise direction, these cells can only hold:

[1,5]–>[4,6]–>[1,6]–>[4,5]–>[4,5]–>[1,6]–>[4,6]–>[1,5]

But there is an even more severe restriction. Cells in the same horizontal row are paired, so if one cell is determined the other must be as well. As a consequence, each puzzle has two, and only two, solutions:

1 –> 4 –> 6 –> 5 –> 4 –> 1 –> 6 –> 5

5 –> 6 –> 1 –> 4 –> 5 –> 6 –> 4 –> 1

Are these to be regarded as two independent, valid solutions to the puzzle? Conventional Sudoku puzzle designers say no. They detest ambiguous loops like this. but I find them interesting. When mangling Sudokus as we have been doing it is not uncommon to find an ambiguous loop of four cells or even six. But this is the first eight-cell ambiguous loop that I have yet encountered.

All this can get very confusing, but a tree diagram can come to the rescue. What we have been doing in Plates 6.4 through 6.10 can be plotted as alternative 1 in Plate 6.11. Choosing cell A2 produces one acceptable solution, **@a**, and one stall, **s**. Breaking the stall with D3 leads to a second acceptable solution, **@b**, and another stall. Finally, breaking that stall yields two similar but different solutions with eight-cell ambiguous loops, **#a** and **#b**. No more stalls result, so the process of analysis has come to an end with pal=3.

How many solutions have been produced by all of this? At first you might say "four", meaning two unique solutions and two ambiguous loops. However, <u>each</u> of the ambiguous loops is actually two answers compressed onto one sheet. The two puzzles differ only in the order of digits around the loop. But they are indeed distinct answers, since each of them contains a different arrangement of 81 digits that satisfy the Sudoku rules and the starting digit set. Hence the Sudoku with a central E5=1 yields <u>six different solutions</u>. No wonder Sudoku designers discard them quickly. Or, they try to fine-tune the puzzle to eliminate multiple answers. In the present case we know by hindsight that merely changing the central digit from 1 to 6 produces a puzzle with only a single answer.

How different would our answers have been if we had chosen different two-digit cells? The answer is shown as alternatives 2 and 3 in Plate 6.11. Alternative 1 produced solution **@a** in the first level of analysis, **@b** in the second, and both ambiguous loops in the third level (pal=3). With alternative 2, ambiguous loop **#a** showed up in the first level, solution **@a** in the second, and **@b** and **#b** in the third level (again, pal=3). Alternative 3 was a tough one. <u>Nothing was learned</u> in the first two search levels. But the third level yielded both solutions **@a** and **@b** in two separate branches, and a fourth level of analysis, or pal=4, was needed to come up with the two ambiguous loops.

These routes to the answers are different, but the answers are the same. The four results, @**a**, @**b**, #**a** and #**b**, are the solutions, and the only solutions, of the starting set with E5=1. I didn't bother printing out the results obtained with alternatives 2 or 3 since we have already seen this same set of answers from alternative 1. Try these other alternatives yourself as practice, and verify that the results are unchanged. With a correct answer or a set of correct answers, the results of any search are always the same; only the pathway to the answers varies. This is in sharp contrast with what happens in a crashed or failed solution, where the particular set of duplicated digits depends on how you carry out the analysis.

All of the branches in the four trees of Plates 5.8 and 6.11 come to an end with either a failure (**x**) an acceptable solution (@) or an ambiguous loop (#). There are no unfinished stalls (**s**). Hence we know that the search is at an end.

Plate 6.12 compares the results obtained with E5=6 (@), and with E5=1 (@**a**, @**b**, #**a**, #**b**). They are identical in 45 of their 81 cells. In molecular evolution, we can make a table or matrix of the number of amino acid differences between molecules of the same protein in different species, and then use these data to build a genealogical or evolutionary family tree of these species. The process is well-established and very dependable (in spite of objections from a few fundamentalists). I tried the same thing with the five "species" from our analysis, **@, @a, @b, #a** and **#b**, but failed to come up with anything significant.

However, the trees that we built in Plate 6.11 are rather similar to the evolutionary trees constructed by molecular biology, with the two choices in a given cell corresponding to mutations in DNA sequences. But there's one dramatic difference. Species in different evolutionary branches of life evolve in different directions, whereas the trees in Plate 6.11 all converge to the very same four solutions. It's rather as if humans on different planets were fated to be exactly the same, whether they evolved from primates or from oysters!

In sum, parallel analysis is a marvelous tool for taking Sudoku puzzles apart, seeing how they are constructed, and then making your own. I find redesigning Sudokus to be far more interesting than merely solving them as they come along, and I hope that you will, also.

Plate 6.1: Starting set A with central digit omitted

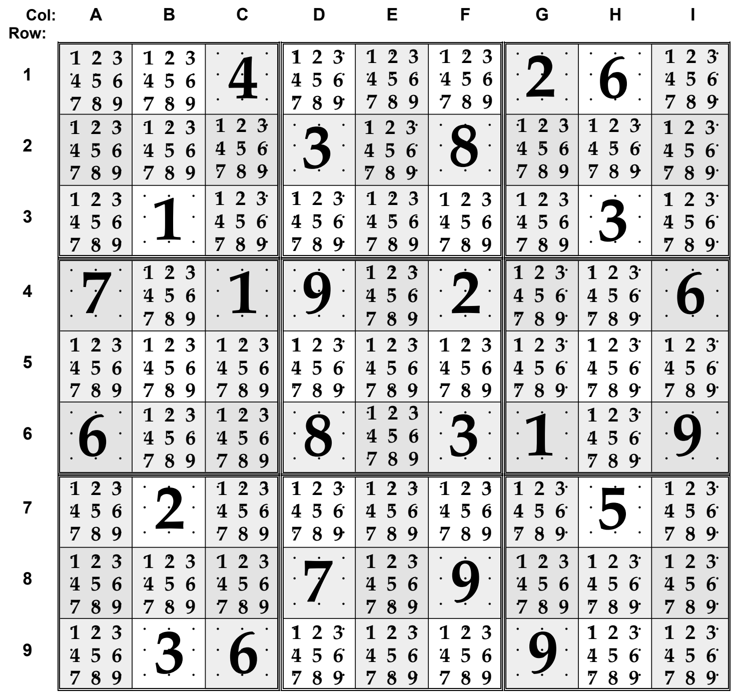


Plate 6.2: After initial crossouts of conflicting digits, B

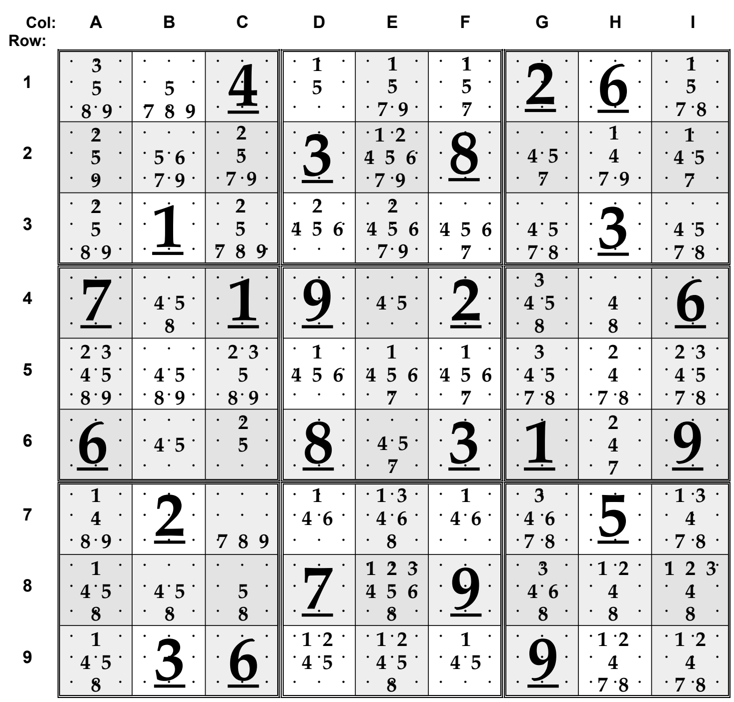


Plate 6.3: Stall, C--taken as far as one can go with basic methods

The central E5 cell can only hold a 1 or a 6

Row	A	B	C	D	E	F	G	H	I
1	3	7	4	1 5	9	1 5	2	6	8
2	2 5	6	2 5	3	4 7	8	4 7	9	1
3	8	1	9	2 4 6	2 6	4 6 7	4 5 7	3	4 5 7
4	7	8	1	9	5	2	3	4	6
5	2 4 5	9	3	1 4 6	1 6	1 4 6 7	5 7 8	2 7 8	2 5 7
6	6	4 5	2 5	8	4 7	3	1	2 7	9
7	9	2	7	1 4 6	1 3 6 8	1 4 6	4 6 8	5	3 4
8	1 4 5	4 5	8	7	1 2 3 6	9	4 6	1 2	2 3 4
9	1 4 5	3	6	1 2 4 5	1 2 8	1 4 5	9	1 2 7 8	2 4 7

Plate 6.4: Introduction of E5=1, and continuation until a new stall

Row	A	B	C	D	E	F	G	H	I
1	3	7	4	1 5	9	1 5	2	6	8
2	2 5	6	2 5	3	4 7	8	4 7	9	1
3	8	1	9	2 4 6	2 6	4 6 7	4 5 7	3	4 5 7
4	7	8	1	9	5	2	3	4	6
5	2 4 5	9	3	4 6	1	4 6 7	5 7 8	2 7 8	2 5 7
6	6	4 5	2 5	8	4 7	3	1	2 7	9
7	9	2	7	1 4 6	3 6 8	1 4 6	4 6 8	5	3 4
8	1 4 5	4 5	8	7	2 3 6	9	4 6	1 2	2 3 4
9	1 4 5	3	6	1 2 4 5	2 8	1 4 5	9	1 2 7 8	2 4 7

Plate 6.5: E5=1, A2=2@a--first acceptable solution

Row \ Col	A	B	C	D	E	F	G	H	I
1	3	7	4	5	9	1	2	6	8
2	2	6	5	3	7	8	4	9	1
3	8	1	9	2	6	4	7	3	5
4	7	8	1	9	5	2	3	4	6
5	4	9	3	6	1	7	5	8	2
6	6	5	2	8	4	3	1	7	9
7	9	2	7	1	3	6	8	5	4
8	5	4	8	7	2	9	6	1	3
9	1	3	6	4	8	5	9	2	7

Plate 6.6: E5=1, A2=5s--stall

Row \ Col	A	B	C	D	E	F	G	H	I
1	3	7	4	1 5	9	1 5	2	6	8
2	5	6	2	3	4	8	7	9	1
3	8	1	9	2 6	2 6	7	4 5	3	4 5
4	7	8	1	9	5	2	3	4	6
5	2	9	3	4 6	1	4 6	5 8	7 8	5 7
6	6	4	5	8	7	3	1	2	9
7	9	2	7	1 4 6	3 6 8	1 4 6	4 8	5	3 4
8	4	5	8	7	2 3	9	6	1	2 3
9	1	3	6	2 4 5	2 8	4 5	9	7 8	2 4 7

Plate 6.7: E5=1, A2=5s, D3=6@b--second acceptable solution

Col: / Row:	A	B	C	D	E	F	G	H	I
1	3	7	4	5	9	1	2	6	8
2	5	6	2	3	4	8	7	9	1
3	8	1	9	6	2	7	4	3	5
4	7	8	1	9	5	2	3	4	6
5	2	9	3	4	1	6	5	8	7
6	6	4	5	8	7	3	1	2	9
7	9	2	7	1	6	4	8	5	3
8	4	5	8	7	3	9	6	1	2
9	1	3	6	2	8	5	9	7	4

Plate 6.8: E5=1, A2=5s, D3=2s--stall

Col: / Row:	A	B	C	D	E	F	G	H	I
1	3	7	4	1 5	9	1 5	2	6	8
2	5	6	2	3	4	8	7	9	1
3	8	1	9	2	6	7	4 5	3	4 5
4	7	8	1	9	5	2	3	4	6
5	2	9	3	4 6	1	4 6	5 8	7 8	5 7
6	6	4	5	8	7	3	1	2	9
7	9	2	7	1 4 6	3 8	1 4 6	4 8	5	3 4
8	4	5	8	7	2 3	9	6	1	2 3
9	1	3	6	4 5	2 8	4 5	9	7 8	2 4 7

Plate 6.9: E5=1, Cs, A2=5s, D3=2s, H5=7#a--first ambiguous solution

Digit cycle can only be 1-4-6-5-4-1-6-5 or 5-6-1-4-5-6-4-1

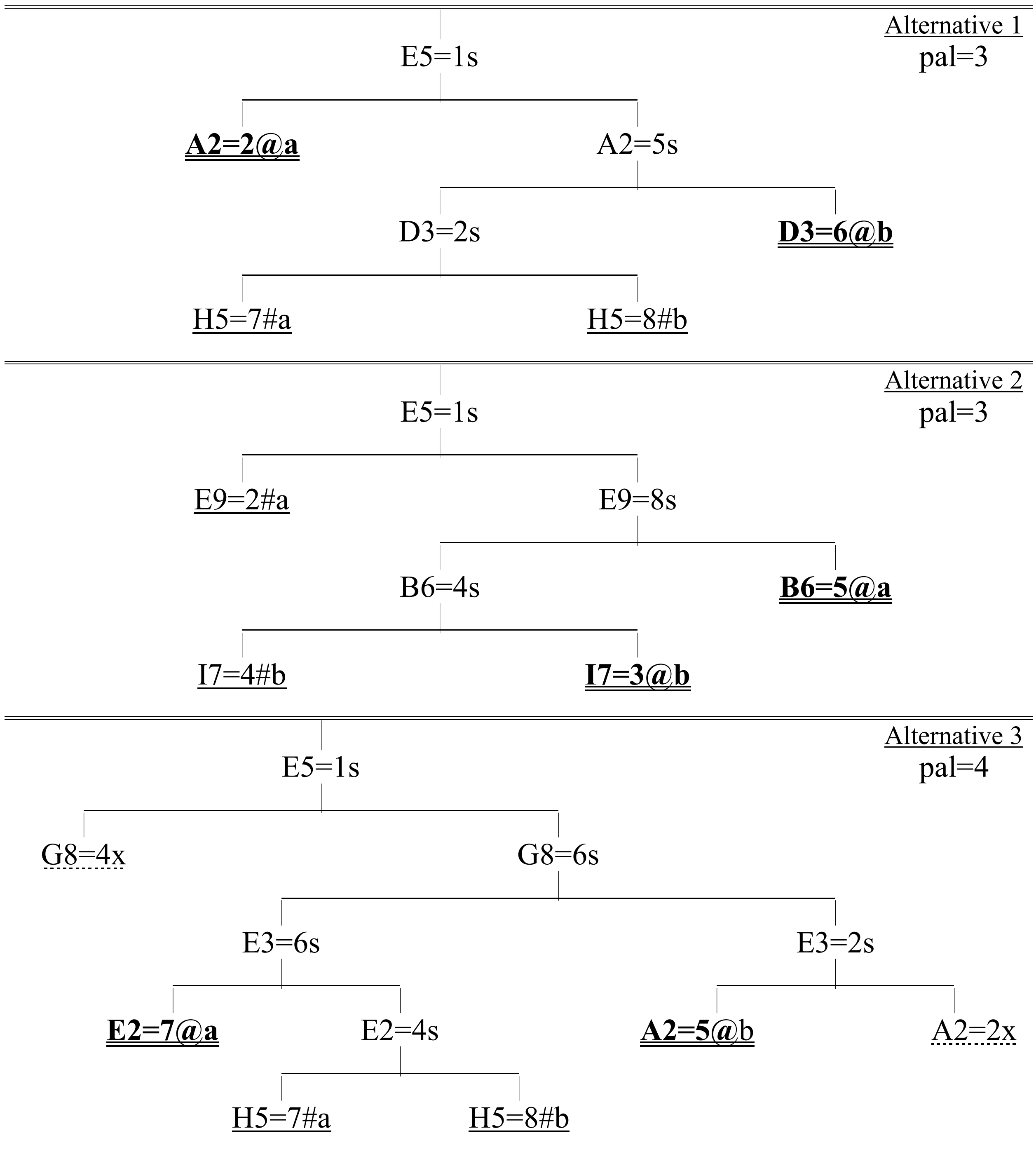
Row: / Col:	A	B	C	D	E	F	G	H	I
1	3	7	4	1 5	9	1 5	2	6	8
2	5	6	2	3	4	8	7	9	1
3	8	1	9	2	6	7	5	3	4
4	7	8	1	9	5	2	3	4	6
5	2	9	3	4 6	1	4 6	8	7	5
6	6	4	5	8	7	3	1	2	9
7	9	2	7	1 6	8	1 6	4	5	3
8	4	5	8	7	3	9	6	1	2
9	1	3	6	4 5	2	4 5	9	8	7

Plate 6.10: E5=1, A2=5s, D3=2s, H5=8#b--second ambiguous solution

Digit cycle can only be 1-4-6-5-4-1-6-5 or 5-6-1-4-5-6-4-1

Row: / Col:	A	B	C	D	E	F	G	H	I
1	3	7	4	1 5	9	1 5	2	6	8
2	5	6	2	3	4	8	7	9	1
3	8	1	9	2	6	7	4	3	5
4	7	8	1	9	5	2	3	4	6
5	2	9	3	4 6	1	4 6	5	8	7
6	6	4	5	8	7	3	1	2	9
7	9	2	7	1 6	3	1 6	8	5	4
8	4	5	8	7	2	9	6	1	3
9	1	3	6	4 5	8	4 5	9	7	2

Plate 6.11: Parallel Solution Trees with E5=1



@ = correct solution
# = closed ambiguous loop, disallowed solution
s = stall, incomplete solution

@a, @b = two different valid solutions
#a, #b = two different ambiguous loops
x = incorrect solution with clashing digits

Plate 6.12: Comparison of results @, @b, @a (above), and #a, #b (below)

45 of 81 cells are invariant.

Col: / Row:	A	B	C	D	E	F	G	H	I
1	3	7	4	5 5 5 / 15 15	9	1 1 1 / 15 15	2	6	8
2	5 5 2 / 5 5	6	2 2 5 / 2 2	3	4 4 7 / 4 4	8	7 7 4 / 7 7	9	1
3	8	1	9	6 6 2 / 2 2	2 2 6 / 6 6	7 7 4 / 7 7	4 4 7 / 5 4	3	5 5 5 / 4 5
4	7	8	1	9	5	2	3	4	6
5	2 2 4 / 2 2	9	3	1 4 6 / 46 46	6 1 1 / 1 1	4 6 7 / 46 46	5 5 5 / 8 5	8 8 8 / 7 8	7 7 2 / 5 7
6	6	4 4 5 / 4 4	5 5 2 / 5 5	8	7 7 4 / 7 7	3	1	2 2 7 / 2 2	9
7	9	2	7	4 1 1 / 16 16	1 6 3 / 8 3	6 4 6 / 16 16	8 8 8 / 4 8	5	3 3 4 / 3 4
8	4 4 5 / 4 4	5 5 4 / 5 5	8	7	3 3 2 / 2 2	9	6	1	2 2 3 / 2 3
9	1	3	6	2 2 4 / 45 45	8 8 8 / 2 8	5 5 5 / 45 45	9	7 7 2 / 8 7	4 4 7 / 7 2