7. OVER THE TOP

"Anything worth doing is worth overdoing." This is a philosophy that can make your life more interesting, if not necessarily more successful. In the last chapter we saw examples of an ambiguous loop having two different solutions. These are considered serious flaws by professional Sudoku designers, and are avoided like the plague. But in fact they can be more interesting than the standard newspaper Sudoku puzzle.

I wasted a lot of time last year in making a systematic study of how many, and how large, ambiguous loops could exist in a solvable Sudoku. The trick was first to map out the loop that you wanted to find, then surround it with digits that satisfy the Sudoku rules, and finally select enough of those digits to permit the puzzle to be solved in the normal way. I named this backward analysis process "Kudosu", and wrote up the results in a small book entitled *Sudoku, Cascades and Kudosu.* In it are giant ambiguous loops made up of as many as 18 cells, each containing the same two digits. I also determined that the maximum number of ambiguous loops for a given Sudoku matrix is four. Plates 7.1–7.4 show a particularly attractive and symmetrical example of a 4-loop Sudoku**. Please do not look at Plates 7.2–7.4 until you have tried to solve Plate 7.1 on your own**. The solution is quite easy; only the basic strategies are needed, in particular the use of ambiguous pairs as in Chapter 3. But the results will astound you.

(Stop reading at this point and try your luck with Plate 7.1. The remainder of this chapter can be considered a "spoiler".)

To pick up where we left off, Plate 7.1 presents the 31 digits of the starting set, along with 3x3 matrices of all nine digits for those cells that are initially undefined. Plate 7.2 shows the results after initial crossouts of forbidden digits in these undefined cells. With very little work, one reaches the final solution in Plate 7.3. But what does this mean: a matrix with eighteen cells containing only the digits 1 and 2? This is Plate 6.9 gone wild.

Plate 7.4 shows what is happening. Instead of one loop with eight cells as in Plate 6.9, we now have three different loops with four cells, and one with six. The three rectangles in dotted or dashed lines are what one usually gets as an ambiguous loop. The central 6-cell loop in solid line is a less common but not especially rare configuration that I call a "bow tie loop".

An interesting question of logic arises. Plate 6.9 had two different solutions, depending on which digit of each cell was picked. How many solutions are there for Plate 7.4? Each of the eighteen cells can hold either a 1 or a 2. Does this mean that once again there are two independent solutions? No, because each of the four loops is independent. You can define a particular solution by specifying the contents of one cell from each of the four loops, for example: A1, D2, G3 and B4. Call these the "marker cells". The various individual solutions then are:

A1	D2	G3	B4
1	1	1	1
1	1	1	2
1	1	2	1
1	1	2	2
1	2	1	1
1	2	1	2
1	2	2	1
1	2	2	2

A1	D2	G3	B4
2	2	2	2
2	2	2	1
2	2	1	2
2	2	1	1
2	1	2	2
2	1	2	1
2	1	1	2
2	1	1	1

This looks like 16 independent solutions, with all possible permutations of 1 and 2 in the four marker cells. With two choices in each of four marker cells, 2 x 2 x 2 x 2 = 16. But the two solutions in each horizontal row above are actually identical. The numbers in a Sudoku have no mathematical significance; they are merely symbols. If you take any Sudoku puzzle and turn every 4 into a 7, and every 7 into a 4, you have changed nothing. The Sudoku is untouched except for names given to some of the symbols. We will come back to this issue in Appendix 1. The conclusion is that there are eight independent solutions to Plate 7.4, not sixteen. Eight is the absolute maximum number of solutions possible for a given Sudoku puzzle, since the greatest number of closed loops that can be present in the 9x9 Sudoku matrix is four. (This is proven in a sequel to this article, entitled *The Sudoku from Hell...An Introduction to Kudosu*.)

I cannot resist showing one final loop Sudoku, with the starting digits seen in Plate 7.5. **As before, stop at this point and see whether you can solve this puzzle on your own.**

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Assuming that you have indeed tried your hand with Plate 7.5, let us continue. Plate 7.6 shows the result of initial crossouts of disallowed digits, and Plate 7.7 is the final answer. Once again we have eighteen cells that are limited to the same two digits (1 and 8), but they are no longer organized into four individual loops. Instead, they all are part of a single, horrendously intricate loop that takes you to every corner of the matrix. Plate 7.8 shows this mega-loop. There is no way that you can construct a closed loop without using all eighteen cells. (Try it and see.)

How many independent solutions are there to this matrix? With all 18 cells connected in a single loop, once you have specified the contents of any one cell you know them all. At first glance Plate 7.8 would seem to have two solutions, one with A1=1, the other with A1=8. But as we have just seen, calling all 1's "eight" and all 8's "one" changes nothing in the Sudoku, and there are no 1's or 8's outside this loop to give the digits identity. So Plate 7.8 has just one independent solution.

What does all this mean to a professional Sudoku puzzle designer? Nothing. These are precisely the kinds of Sudokus that the professional designer rejects. But to me this rearranging and modification of Sudokus is much more interesting than merely passively solving them.

Plate 7.1 Starting digit set, A

Col: / Row:	A	B	C	D	E	F	G	H	I
1	1 2 3 4 5 6 7 8 9	3	1 2 3 4 5 6 7 8 9	4	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9
2	7	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	4	1 2 3 4 5 6 7 8 9	6
3	1 2 3 4 5 6 7 8 9	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	3	1 2 3 4 5 6 7 8 9
4	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	4	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	7
5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	4	1 2 3 4 5 6 7 8 9	7	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
6	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	3	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	4
7	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	4	1 2 3 4 5 6 7 8 9
8	3	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	5
9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	3	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9

Plate 7.2: After initial crossouts, B

Col: / Row:	A	B	C	D	E	F	G	H	I
1	1 2	3	1 2	4	1 2 5 9	6	1 2 5 7 9	8	1 2 9
2	7	1 2 8	9	1 2 5	1 2 3 5 8	1 2 5	4	1 2 5	6
3	1 2 4	5	1 2 6	1 2 7	1 2 8 9	1 2 7 9	1 2 7 9	3	1 2 9
4	8	1 2	1 2 3 5	9	1 2 5 6	4	1 2 3 5	1 2 5 6	7
5	1 2 5 9	1 2	4	1 2 5 6	7	1 2 5	8	1 2 5 6	1 2 3 9
6	6	1 2 7	1 2 5 7 9	8	1 2 5	3	1 2 5 9	1 2 5	4
7	1 2 5	9	1 2 5 7	1 2 5 6 7	1 2 5 6	1 2 5 7	1 2 3 7	4	1 2 3 8
8	3	1 2 4 7	8	1 2 7	1 2 4 9	1 2 7 9	6	1 2 7	5
9	1 2 4 5	6	1 2 5 7	3	1 2 4 5	8	1 2 7	9	1 2

Plate 7.3: Solution, C

Col: / Row:	A	B	C	D	E	F	G	H	I
1	1 2	3	1 2	4	5	6	7	8	9
2	7	8	9	1 2	3	1 2	4	5	6
3	4	5	6	7	8	9	1 2	3	1 2
4	8	1 2	3	9	1 2	4	5	6	7
5	9	1 2	4	6	7	5	8	1 2	3
6	6	7	5	8	1 2	3	9	1 2	4
7	1 2	9	1 2	5	6	7	3	4	8
8	3	4	8	1 2	9	1 2	6	7	5
9	5	6	7	3	4	8	1 2	9	1 2

Plate 7.4: Three 4-cell loops and one 6-cell

Col: / Row:	A	B	C	D	E	F	G	H	I
1	1 2	3	1 2	4	5	6	7	8	9
2	7	8	9	1 2	3	1 2	4	5	6
3	4	5	6	7	8	9	1 2	3	1 2
4	8	1 2	3	9	1 2	4	5	6	7
5	9	1 2	4	6	7	5	8	1 2	3
6	6	7	5	8	1 2	3	9	1 2	4
7	1 2	9	1 2	5	6	7	3	4	8
8	3	4	8	1 2	9	1 2	6	7	5
9	5	6	7	3	4	8	1 2	9	1 2

Plate: 7.5 Starting digit set, A

Col: / Row:	A	B	C	D	E	F	G	H	I
1	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	4	3	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
2	5	1 2 3 4 5 6 7 8 9	4	3	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	2	7
3	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	2	1 2 3 4 5 6 7 8 9	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
4	2	7	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	3	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9
5	6	1 2 3 4 5 6 7 8 9	3	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	2	1 2 3 4 5 6 7 8 9	5
6	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	2	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	4	3
7	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	3	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
8	4	3	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	2	7	1 2 3 4 5 6 7 8 9	6
9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	2	7	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9

Plate: 7.6: After initial crossouts, B

Col: / Row:	A	B	C	D	E	F	G	H	I
1	1 7 8	1 2 8	1 7 8	1 5 8 9	6	4	3	1 8	1 8 9
2	5	1 6 8	4	3	1 8	1 8 9	1 6 8	2	7
3	1 3 7 8	1 6 8	9	1 8	2	1 7 8	5	1 6 8	1 4 8
4	2	7	1 5 8	1 4 5 6 8	1 4 8	3	1 6 8	9	1 8
5	6	1 4 8	3	1 4 8	9	1 7 8	2	1 7 8	5
6	1 8	9	1 5 8	2	1 7 8	1 5 6 7 8	1 6 8	4	3
7	1 7 8	1 5 8	6	1 4 8	3	1 8	9	1 5 8	1 2 4 8
8	4	3	1 5 8	1 8 9	1 8	2	7	1 5 8	6
9	1 8 9	1 8	2	7	5	1 6 8 9	1 4 8	1 3 8	1 4 8

Plate: 7.7: Solution, C

Col: / Row:	A	B	C	D	E	F	G	H	I
1	1 8	2	7	5	6	4	3	1 8	9
2	5	6	4	3	1 8	9	1 8	2	7
3	3	1 8	9	1 8	2	7	5	6	4
4	2	7	5	6	4	3	1 8	9	1 8
5	6	4	3	1 8	9	1 8	2	7	5
6	1 8	9	1 8	2	7	5	6	4	3
7	7	5	6	4	3	1 8	9	1 8	2
8	4	3	1 8	9	1 8	2	7	5	6
9	9	1 8	2	7	5	6	4	3	1 8

Plate: 7.8: One giant convoluted 18-cell loop